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Weak boundedness of Calderón–Zygmund operators on noncommutative L_1 -spaces



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ABSTRACT

In 2008, J. Parcet showed the $(1,1)$ weak-boundedness of Calderón–Zygmund operators acting on functions taking values in a von Neumann algebra. We propose a simplified version of his proof using the same tools: Cuculescu’s projections and a pseudo-localisation theorem. This will enable us to recover the L_p -boundedness of Calderón–Zygmund operators with Hilbert valued kernels acting on operator valued functions for $1 < p < \infty$ and an L_p -pseudo-localisation result of P. Hytönen.

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1. Introduction

Before properly introducing the topic of this paper, it should be said that our main purpose is to offer a simplified version of the argument presented by J. Parcet in [9]. Consequently, its interest lies in the shorter and clearer proof it provides. We believe it can be used to expand the reach of the theorem as shown by the slight improvement we make to its hypotheses and an application we present.

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Our main result belongs to the now well developed theory of singular integrals. The latter was initiated in the 1950's by Calderón and Zygmund who found a very useful sufficient condition for a kernel operator to be bounded on L_p for $1 < p < \infty$. It can be expressed (without details about definition) by the following theorem:

Theorem 1.1. *Let $n \in \mathbb{N}$, $p \in (1, \infty)$ and k a measurable function from \mathbb{R}^{2n} to \mathbb{C} verifying the size and smoothness conditions. Then if the formal expression:*

$$Tf(x) = \int_{y \in \mathbb{R}^n} k(x, y) f(y) dy$$

defines a bounded operator T on $L_2(\mathbb{R}^n)$, it also defines a bounded operator (still noted T) on $L_p(\mathbb{R}^n)$. T is called a Calderón–Zygmund operator and k its kernel.

Size and smoothness will be defined later in a more general context. For the $p = 1$ case, only weak boundedness is true in general:

Theorem 1.2. *With the same conditions and notations, T defines an operator on $L_1(\mathbb{R}^n)$ and there exists a constant C such that for all $f \in L_1(\mathbb{R}^n)$:*

$$\sup_{t>0} t\mu\{Tf > t\} \leq C\|f\|_1$$

where μ is the Lebesgue measure.

A motivation to show this second theorem is that it directly implies the first one by real interpolation and duality.

The ideas behind these two theorems remain valid for kernels and functions taking values in different vector spaces, which makes them a great tool to show boundedness of certain operators such as generalized Hilbert transform or Littlewood–Paley inequalities. What we prove in this paper is a generalisation of the second theorem to noncommutative integration. The result has already proven to be useful since it is the main ingredient used in [1] and [14] in which it is shown that there exists a constant c such that for any Lipschitz function f and for any self-adjoints operators x and y ,

$$\|f(x) - f(y)\|_{1,\infty} \leq c\|f'\|_\infty\|x - y\|_1.$$

This inequality does not directly express the weak $(1, 1)$ boundedness of a Calderón–Zygmund operator but is reduced to it in the mentioned papers. We also hope that our main result is a way to tackle generalisations of classical inequalities on L_p whose proof relies on Calderón–Zygmund theory. Another approach for this kind of problem is to show BMO boundedness rather than weak boundedness and conclude thanks to interpolation theory. This strategy is often easier. It first appears in [7] to show the boundedness of some Fourier multipliers. And has also been applied in [15], where L_p -boundedness of

Calderón–Zygmund operators with operator valued kernels and column valued functions is used to study Hardy spaces on quantum tori as well as in [6] where it is applied to fully noncommutative Calderón–Zygmund operators, in quantum Euclidean spaces.

An introduction to noncommutative L_p -spaces can be found in [13]. We will only briefly recall some basic definitions and results. A noncommutative measure space is a von Neumann algebra \mathcal{M} equipped with a semifinite normal faithful trace τ . For all $x \in \mathcal{M}$, we can define by the functional calculus:

$$\|x\|_p = \tau(|x|^p)^{1/p}.$$

Denote $S_p = \{x \in \mathcal{M} : \|x\|_p < \infty\}$ and $L_p(\mathcal{M}) = \overline{(S_p, \|\cdot\|_p)}$. The elements of $L_p(\mathcal{M})$ can be identified with unbounded operator affiliated to \mathcal{M} . A large part of classical integration theory still holds in this context such as Hölder's inequality, duality and interpolation. In particular, we will need the noncommutative concept of martingales. First, a *filtration* on \mathcal{M} is an increasing sequence $(\mathcal{M}_n)_{n \in \mathbb{N}}$ of von Neumann subalgebras of \mathcal{M} with weak- \star dense union and such that τ restricted to each \mathcal{M}_n remains semifinite. This guarantees the existence of conditional expectations \mathcal{E}_n on \mathcal{M}_n which extend to contractions from $L_p(\mathcal{M})$ to $L_p(\mathcal{M}_n)$ for all $p \geq 1$. With this in mind, the definition of martingale is straightforward.

1.1. Main theorem

We will now introduce the notations that will allow us to state the main theorem of this paper. Let $(\widetilde{\mathcal{M}}, \tau_{\widetilde{\mathcal{M}}})$ be a noncommutative measure space and \mathcal{M} a von Neumann subalgebra of $\widetilde{\mathcal{M}}$ such that $\tau_{\widetilde{\mathcal{M}}}$ restricted to \mathcal{M} (denoted $\tau_{\mathcal{M}}$) is semifinite. $L_p(\mathcal{M})$ is naturally included in $L_p(\widetilde{\mathcal{M}})$ for all $1 \leq p \leq \infty$. Let \mathcal{M}' be the commutant of \mathcal{M} . We will consider \mathcal{N} and $\widetilde{\mathcal{N}}$ the von Neumann algebras of $*$ -weakly measurable \mathcal{M} - and $\widetilde{\mathcal{M}}$ -valued functions on \mathbb{R}^n i.e. the von Neumann tensor products $\mathcal{M} \overline{\otimes} L_\infty(\mathbb{R}^n)$ and $\widetilde{\mathcal{M}} \overline{\otimes} L_\infty(\mathbb{R}^n)$, τ will denote their natural trace (with no ambiguity since \mathcal{N} is naturally included in $\widetilde{\mathcal{N}}$). Let T be a Calderón–Zygmund operator associated with a kernel $k : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathcal{M}' \cap \widetilde{\mathcal{M}}$, formally given by the expression:

$$Tf(x) = \int_{y \in \mathbb{R}^n} k(x, y) f(y) dy$$

Definition 1.3. Say that T has Lipschitz parameter γ ($0 < \gamma \leq 1$) if for all x, y and z in \mathbb{R}^n verifying $|x - z| \leq \frac{1}{2}|y - z|$, the following *smoothness estimates* hold:

$$\begin{aligned} \|k(x, y) - k(z, y)\|_{\widetilde{\mathcal{M}}} &\leq \frac{|x - z|^\gamma}{|y - z|^{n+\gamma}}, \\ \|k(y, x) - k(y, z)\|_{\widetilde{\mathcal{M}}} &\leq \frac{|x - z|^\gamma}{|y - z|^{n+\gamma}}. \end{aligned}$$

We will also suppose that T verifies the *size condition*, for all $x, y \in \mathbb{R}^n$:

$$\|k(x, y)\|_{\widetilde{\mathcal{M}}} \leq \frac{1}{|x - y|^n}.$$

Remark 1.4. This size condition will never appear throughout the proof. It is however used in Remark 1.6. Moreover, it is implied by the smoothness condition provided that k goes to 0 at infinity.

On the contrary, the smoothness condition will be crucial to many computations. A Hörmander type condition:

$$\sup_{x \in \mathbb{R}^n} \int_{|x-y| > 2|x-x'|} |k(x, y) - k(x', y)| dy < C$$

would not suffice to use the ideas of this paper. We would at least need a decay of the following type:

$$\sup_{x \in \mathbb{R}^n} \int_{|x-y| > \alpha|x-x'|} |k(x, y) - k(x', y)| dy < \frac{C}{\alpha^n}, \quad \alpha > 2.$$

But we do not know if it is a strong enough condition since part of the proof of pseudo-localisation relies on pointwise estimates of the kernel.

Define, for all $t > 0$ and $f \in \widetilde{\mathcal{N}}$:

$$\lambda_t(f) = \tau(\{f > t\}).$$

The main theorem we are aiming to prove in this article is the following.

Theorem 1.5. *There exists a constant $c_{n,\gamma}$ depending only on n and γ such that for all Calderón–Zygmund operator T with $(\mathcal{M}' \cap \widetilde{\mathcal{M}})$ -valued kernel and Lipschitz parameter $0 < \gamma \leq 1$, verifying the size estimate and bounded from $L_2(\mathcal{N})$ to $L_2(\widetilde{\mathcal{N}})$, for all $f \in L_1(\mathcal{N})$:*

$$\sup_{t>0} t\lambda_t(Tf) \leq c_{n,\gamma} \|f\|_1$$

To prove this theorem, we will need a pseudo-localisation result which constitutes the first part of this paper. This is where the most important simplifications are made compared to J. Parcet's work. In particular, our proof is elementary and does not explicitly use the size condition. We decide to directly show the result with noncommutative variables but the pseudo-localisation theorem is essentially a commutative one. The second part, which is the proof of the main theorem has been shortened and clarified thanks to

more efficient organisation and computations but the underlying ideas all appear in [9]. In particular, we use the same decomposition which relies on Cuculescu's projections. It is a natural noncommutative counterpart to the decomposition used for the classical proof but unfortunately, new kind of terms appear which will necessitate more refined estimates and in particular, the pseudo-localisation theorem. In a third section, we show a boundedness result for singular integrals with Hilbert valued kernels and operator valued functions. It was already shown in [8] by J. Parcet and T. Mei but follows directly from Theorem 1.5 and Khintchine inequalities. We conclude this paper by providing an L_p -pseudo-localisation theorem similar to the one of P. Hytönen in [5]. We would like to thank the referee for bringing the latter to our attention. The result follows mainly from our proof of L_2 -pseudo-localisation, martingale inequalities and interpolation.

1.2. A technical remark and notations

The following remark allows us to manipulate the integral expression of T , in particular to use the smoothness condition.

Remark 1.6. For any Calderón–Zygmund operator T there exists a Calderón–Zygmund operator T' such that $Tf = T'f + gf$ for all f in $L_2(\mathcal{N})$ and T' is the strong limit in $\mathcal{B}(L_2(\mathcal{N}))$ of operators T_i given by:

$$T_i f(x) = \int_{y \in \mathbb{R}^n} k_i(x, y) f(y) dy,$$

for any $f \in L_2(\mathcal{N})$. Here, k_i can be taken as a truncation of k so that the integral makes sense and g is a bounded function in $\tilde{\mathcal{N}}$. This fact is explained in more details in [4], proposition 8.1.11. As a consequence, it suffices to prove the theorem for operators given by a converging integral formula on $L_2(\mathcal{N})$.

The proof of the main theorem will rely on the use of dyadic martingales which will require a few notations.

- Since we deal with cubes, the ∞ -norm will be easier to manipulate than the Euclidean one. Hence, for all $x \in \mathbb{R}^n$, we set $|x| = \|x\|_\infty$ in the remainder of the paper.
- \mathcal{Q} will denote the set of all dyadic cubes and \mathcal{Q}_k the set of dyadic cubes of edge length 2^{-k} . Let $V_k = 2^{-nk}$ be the volume of such a cube.
- For all x in \mathbb{R}^n , $Q_{x,k}$ will denote the cube in \mathcal{Q}_k containing x and $c_{x,k}$ its centre.
- Let $(\mathcal{E}_k)_{k \in \mathbb{Z}}$ be the martingale associated with dyadic filtration, i.e. for all f :

$$\mathcal{E}_k(f)(x) = \frac{1}{V_k} \int_{Q_{x,k}} f(t) dt$$

For convenience, we will write $f_k := \mathcal{E}_k(f)$ and $\Delta_k(f) := f_k - f_{k-1} =: df_k$. The filtration associated with these expectations will be denoted by $(\mathcal{N}_k)_{k \in \mathbb{Z}}$ where \mathcal{N}_k is the von Neumann subalgebra of \mathcal{N} constituted of functions that are constant on dyadic cubes of edge length 2^{-k} .

- For any odd positive integer i and Q in \mathcal{Q}_k , iQ will designate the image of Q by the homothety of centre c_Q and parameter i such that iQ is the union of i^n cubes in \mathcal{Q}_k .
- Notice that for all $x, y \in \mathbb{R}^n$ and $k \in \mathbb{Z}$, $x \in iQ_{y,k} \Leftrightarrow y \in iQ_{x,k}$.

The notation $A \lesssim B$ will stand for “there exists a constant c depending only on n and γ such that $A \leq cB$ ”.

In the next section, we will frequently use “polar” changes of coordinates with respect to the norm $|\cdot|$ since it is more adapted to our problem. The spherical element of volume is replaced by the border of a cube which leads to a similar formula:

$$\int_{\mathbb{R}^n} f(|x|) dx = \int_{\mathbb{R}^+} 2n(2r)^{n-1} f(r) dr.$$

2. Pseudo-localisation

2.1. Theorem

A localisation result would be of the form $\text{supp } Tf \approx \text{supp } f$ which is ideal to show weak boundedness. The theorem that follows expresses in a way the fact that singular integrals rapidly vanish outside of the support of the function to which they are applied. A simpler result of this type appears in the commutative proof in the L_1 -context and is enough to conclude in this case. But, as mentioned before, the noncommutative case requires new tools such as the L_2 -version of pseudo-localisation that follows. Note that the proof is written with operator-valued functions because we will need this result later but is almost a copy of the proof we had with scalar functions. So, surprisingly, the most technical part of the proof of our main theorem is purely commutative.

Theorem 2.1. *Let $f \in L_2(\mathcal{N})$ and $s \in \mathbb{N}$. For all $k \in \mathbb{Z}$, let A_k and B_k be projections in \mathcal{N}_k such that $A_k^\perp df_{k+s} = df_{k+s} B_k^\perp = 0$. For any odd positive integer d , write:*

$$A_k = \bigvee_{Q \in \mathcal{Q}_k} A_Q \mathbf{1}_Q, \quad A_Q \in \mathcal{M} \text{ and define } dA_k := \bigvee_{Q \in \mathcal{Q}_k} A_Q \mathbf{1}_{dQ}.$$

Define dB_k the same way. Let:

$$A_{f,s} := \bigvee_{k \in \mathbb{Z}} 5A_k \text{ and } B_{f,s} := \bigvee_{k \in \mathbb{Z}} 5B_k. \quad (1)$$

Let T be a Calderón–Zygmund operator associated with a kernel k with Lipschitz parameter γ , verifying the size condition, taking values in $\mathcal{M}' \cap \widetilde{\mathcal{M}}$ and bounded from $L_2(\mathcal{N})$ to $L_2(\widetilde{\mathcal{N}})$. Then for all $s \in \mathbb{N}$ and $f \in L_2(\mathcal{N})$ we have:

$$\|A_{f,s}^\perp(Tf)\|_2 \lesssim 2^{-\frac{\gamma s}{2}} \|f\|_2 \text{ and } \|(Tf)B_{f,s}^\perp\|_2 \lesssim 2^{-\frac{\gamma s}{2}} \|f\|_2.$$

Throughout the course of the proof we will often use the following lemma:

Lemma 2.2 (Schur). *Let T be an operator on $L_2(\mathcal{N})$ given by a $\widetilde{\mathcal{M}}$ -valued kernel:*

$$Tf(x) = \int_{\mathbb{R}^n} k(x, y) f(y) dy.$$

Let $S_1(x) = \int_{\mathbb{R}^n} \|k(x, y)\|_{\widetilde{\mathcal{M}}} dy$ and $S_2(y) = \int_{\mathbb{R}^n} \|k(x, y)\|_{\widetilde{\mathcal{M}}} dx$ then:

$$\|T\|_{\mathcal{B}(L_2(\mathcal{N}))} \leq \sqrt{\|S_1\|_\infty \|S_2\|_\infty}.$$

Proof. This is not different from the commutative case (see [9]). Let $f \in L_2(\mathcal{N})$:

$$\begin{aligned} \|Tf\|_2^2 &= \int_{\mathbb{R}^n} \left\| \int_{\mathbb{R}^n} k(x, y) f(y) dy \right\|_2^2 dx \\ &\leq \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \|k(x, y) f(y)\|_2 dy \right)^2 dx \\ &\leq \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \|k(x, y)\|_{\widetilde{\mathcal{M}}} \|f(y)\|_2 dy \right)^2 dx \\ &\leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \|k(x, y)\|_{\widetilde{\mathcal{M}}} dy \int_{\mathbb{R}^n} \|k(x, y)\|_{\widetilde{\mathcal{M}}} \|f(y)\|_2^2 dy dx \\ &\leq \|S_1\|_\infty \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \|k(x, y)\|_{\widetilde{\mathcal{M}}} \|f(y)\|_2^2 dy dx \\ &\leq \|S_1\|_\infty \|S_2\|_\infty \int_{\mathbb{R}^n} \|f(y)\|_2^2 dy = \|S_1\|_\infty \|S_2\|_\infty \|f\|_2^2 \quad \square \end{aligned}$$

It will also be important to note that by construction, for all $x, y \in \mathbb{R}^n$ such that $x \in 5Q_{k,y}$, we have $A_k(y) \leq (5A_k)(x) \leq A_{f,s}(x)$ and $A_k^\perp(y) df_{k+s}(y) = 0$. Consequently:

$$A_{f,s}^\perp(x) df_{k+s}(y) = (5A_k)^\perp(x) df_{k+s}(y) = 0. \quad (2)$$

We will only show the pseudo-localisation theorem in the case of left multiplications since the exact same proof can be written for right multiplication. Another way of seeing

it is that left and right multiplication are equivalent by taking adjoints. The general strategy will be to find an operator T' verifying $A_{f,s}^\perp T f = A_{f,s}^\perp T' f$ and such that we can control $\|T'\|_{\mathcal{B}(L_2(\mathcal{N}))}$ thanks to the lemma above. By [Remark 1.6](#), we can also suppose that the integral defining T converges. The result for any T follows then directly by approximations.

2.2. The s shift

Let $f \in L_2(\mathcal{N})$, fixed throughout the proof. Let $s \in \mathbb{N}$. To make use of the construction of $A_{f,s}$ we immediately write $T = \sum_{k \in \mathbb{Z}} T \Delta_{k+s}$ where the sum converges for the strong operator topology on $\mathcal{B}(L_2(\mathcal{N}))$. We will show in this section that the constant $2^{-\gamma s}$ appears quite easily thanks to the smoothness condition when estimating the norm of $A_{f,s}^\perp T \Delta_{k+s} f$. The tricky part will be to glue these pieces back together in the following sections. Let $k \in \mathbb{Z}$, note that:

$$\int_{\mathbb{R}^n} k(x, c_{y,k+s-1}) df_{k+s}(y) dy = \int_{\mathbb{R}^n} \mathcal{E}_{k+s-1}(k(x, c_{\cdot, k+s-1}) df_{k+s}(\cdot))(y) dy = 0.$$

Therefore we can write:

$$\begin{aligned} A_{f,s}^\perp(x) T df_{k+s}(x) &= A_{f,s}^\perp(x) \int_{\mathbb{R}^n} k(x, y) df_{k+s}(y) dy \\ &= A_{f,s}^\perp(x) \int_{\mathbb{R}^n} (k(x, y) - k(x, c_{y,k+s-1})) df_{k+s}(y) dy \end{aligned}$$

$A_k^\perp(y)$ commutes with $k(x, y)$ and $k(x, c_{y,k+s-1})$ since k is \mathcal{M}' -valued so:

$$A_{f,s}^\perp(x) T df_{k+s}(x) = A_{f,s}^\perp(x) \int_{\mathbb{R}^n} (k(x, y) - k(x, c_{y,k+s-1}))(5A_k)^\perp(x) df_{k+s}(y) dy$$

and by [\(2\)](#):

$$= A_{f,s}^\perp(x) \int_{\mathbb{R}^n} \mathbf{1}_{x \notin 5Q_{k,y}} (k(x, y) - k(x, c_{y,k+s-1})) df_{k+s}(y) dy.$$

Denote by T_k the operator associated with the kernel

$$k_k : (x, y) \mapsto \mathbf{1}_{x \notin 5Q_{k,y}} k(x, y) \tag{3}$$

and $T_{k,s}$ the operator associated with the kernel

$$k_{k,s} : (x, y) \mapsto \mathbf{1}_{x \notin 5Q_{k,y}} (k(x, y) - k(x, c_{y,k+s-1})). \tag{4}$$

It will be useful to express the result of the previous computation in terms of these operators:

Lemma 2.3. *For all $k \in \mathbb{Z}$ and $s \in \mathbb{N}$:*

$$A_{f,s}^\perp T \Delta_{k+s} f = A_{f,s}^\perp T_k \Delta_{k+s} f = A_{f,s}^\perp T_{k,s} \Delta_{k+s} f, \quad (5)$$

and more precisely:

$$T_k \Delta_{k+s} f = T_{k,s} \Delta_{k+s} f. \quad (6)$$

The introduction of $T_{k,s}$ is motivated by the following lemma.

Lemma 2.4. *For all $k \in \mathbb{Z}$ and $s \in \mathbb{N}$:*

$$\|T_{k,s}\|_{\mathcal{B}(L_2(\mathcal{N}))} \lesssim 2^{-s\gamma}. \quad (7)$$

Proof. The smoothness hypothesis on T (see Definition 1.3) gives:

$$\begin{aligned} \|k_{k,s}(x, y)\|_{\widetilde{\mathcal{M}}} &= \mathbf{1}_{x \notin 5Q_{k,y}} \|k(x, y) - k(x, c_{y,k+s-1})\|_{\widetilde{\mathcal{M}}} \\ &\leq \mathbf{1}_{x \notin 5Q_{k,y}} \frac{|y - c_{y,k+s-1}|^\gamma}{|y - x|^{n+\gamma}} \\ &\lesssim \mathbf{1}_{x \notin 5Q_{k,y}} \frac{2^{-(k+s)\gamma}}{|y - x|^{n+\gamma}}. \end{aligned} \quad (8)$$

The condition $|y - c_{y,k+s-1}| \leq \frac{1}{2}|x - y|$ is verified even for $s = 0$ as long as $x \notin 5Q_{k,y}$. This estimate is enough to apply Schur's lemma and we obtain the result by a direct computation using a “polar” change of coordinates. \square

2.3. One more cancellation property

We introduce another kernel modification whose purpose is not immediately clear but will be crucial in obtaining the estimates to apply Schur's lemma later.

Proposition 2.5. *For all $k \in \mathbb{Z}$ and $s \in \mathbb{N}$ there exists an operator $S_{k,s}$ associated with a kernel $s_{k,s}$ such that the three following conditions hold:*

1. $A_{f,s}^\perp T \Delta_{k+s} f = A_{f,s}^\perp S_{k,s} \Delta_{k+s} f$,
2. $\int_{y \in \mathbb{R}^n} s_{k,s}(x, y) dy = 0$ for all $x \in \mathbb{R}^n$,
3. $\|s_{k,s}(x, y)\|_{\widetilde{\mathcal{M}}} \lesssim \mathbf{1}_{|x-y| > 2^{-k-1}} \frac{2^{-(k+s)\gamma}}{|y - x|^{n+\gamma}}.$

Proof. Consider $R_{k,s}$ an operator associated with the kernel $r_{k,s}$ defined by:

$$r_{k,s}(x, y) := \mathbf{1}_{\mathcal{A}_k} \frac{K(x) 2^{-(k+s)\gamma}}{I_{n,\gamma} |y - x|^{n+\gamma}}$$

where $\mathcal{A}_k := \{(x, y) : x \in 5Q_{y,k}, x \notin 3Q_{y,k}\}$, $K(x) = -\int_{\mathbb{R}^n} k_{k,s}(x, t) dt$ and $I_{n,\gamma} = \int_{\mathbb{R}^n} \mathbf{1}_{\mathcal{A}_k}(0, t) \frac{2^{-(k+s)\gamma}}{|t|^{n+\gamma}} dt$.

We claim that $S_{k,s} := T_{k,s} + R_{k,s}$ satisfies the conditions above.

Condition 1. $A_{f,s}^\perp R_{k,s} \Delta_{k+s} f = 0$ is verified since:

$$\text{supp } r_{k,s} \subset \{(x, y) : x \in 5Q_{y,k}\} \text{ i.e. } r_{k,s} = \mathbf{1}_{x \in 5Q_{y,k}} r_{k,s}.$$

Indeed, the shift section shows that to compute $A_{f,s}^\perp R_{k,s} \Delta_{k+s} f$, and in particular $(5A_k)^\perp R_{k,s} \Delta_{k+s} f$, we might as well replace the kernel $r_{k,s}$ by $\mathbf{1}_{x \notin 5Q_{k,y}} r_{k,s} = \mathbf{1}_{x \notin 5Q_{k,y}} \mathbf{1}_{x \in 5Q_{y,k}} r_{k,s} = 0$.

Condition 2. It is direct by definition of K and $I_{n,\gamma}$. Let $x \in \mathbb{R}^n$:

$$\int_{y \in \mathbb{R}^n} s_{k,s}(x, y) dy = \int_{\mathbb{R}^n} k_{k,s}(x, y) dy + K(x) = 0.$$

Condition 3. It suffices to show that $\|K(x)\|_{\widetilde{\mathcal{M}}}$ is bounded by a constant depending only on n and γ . By a “polar” change of coordinates, there exists a constant c_n such that:

$$\|K(x)\|_{\widetilde{\mathcal{M}}} \leq \int_{|t| > 4 \cdot 2^{-k}} \frac{2^{-(k+s)\gamma}}{|t|^{n+\gamma}} dt = c_n 2^{-\gamma s}.$$

The bound does not depend on x , as we needed. \square

2.4. The decomposition

We are ready to decompose T into operators whose norms in $\mathcal{B}(L_2(\mathcal{N}))$ are controlled. Fix $s \in \mathbb{N}$, write:

$$\begin{aligned} A_{f,s}^\perp T f &= A_{f,s}^\perp \sum_{k \in \mathbb{Z}} T \Delta_{k+s} f = A_{f,s}^\perp \sum_{k \in \mathbb{Z}} (T_k + R_{k,s}) \Delta_{k+s} f \\ &= A_{f,s}^\perp \sum_{k \in \mathbb{Z}} \sum_{i \in \mathbb{Z}} \Delta_{k+i} (T_k + R_{k,s}) \Delta_{k+s} f \end{aligned}$$

Note that for $i \geq 1$, $(5A_k)^\perp$ commutes with Δ_{k+i} and recall that the first point of $R_{k,s}$ ’s construction implies that $(5A_k)^\perp R_{k,s} \Delta_{k+s} f = 0$, then:

$$\begin{aligned} A_{f,s}^\perp \Delta_{k+i} (T_k + R_{k,s}) \Delta_{k+s} f &= A_{f,s}^\perp \Delta_{k+i} (5A_k)^\perp (T_k + R_{k,s}) \Delta_{k+s} f \\ &= A_{f,s}^\perp \Delta_{k+i} T_k \Delta_{k+s} f \end{aligned}$$

For any $i \in \mathbb{Z}$, we have:

$$\begin{aligned} A_{f,s}^\perp \Delta_{k+i} (T_k + R_{k,s}) \Delta_{k+s} f &= A_{f,s}^\perp \Delta_{k+i} (T_{k,s} + R_{k,s}) \Delta_{k+s} f \\ &= A_{f,s}^\perp \Delta_{k+i} S_{k,s} \Delta_{k+s} f \end{aligned}$$

Now we can write our final decomposition:

$$A_{f,s}^\perp T f = A_{f,s}^\perp \left(\sum_{i=0}^{\infty} \Phi_i f + \sum_{i=1}^{\infty} \Psi_i f \right) \quad (9)$$

where:

$$\Phi_i = \sum_{k \in \mathbb{Z}} \Delta_{k+i} T_k \Delta_{k+s} \text{ and } \Psi_i = \sum_{k \in \mathbb{Z}} \Delta_{k-i} S_{k,s} \Delta_{k+s}$$

We have then reduced our pseudo-localisation theorem to the following proposition.

Proposition 2.6. *The following estimates hold for $i \geq 0$:*

$$\|\Phi_i\|_{\mathcal{B}(L_2(\mathcal{N}))} \lesssim 2^{-\gamma \frac{i+s}{2}} \text{ and } \|\Psi_i\|_{\mathcal{B}(L_2(\mathcal{N}))} \lesssim \sqrt{1+i} 2^{-\gamma \frac{i+s}{2}}$$

The proof of this result will occupy the next two parts.

2.5. Estimate for Φ_i

Let $i \geq 0$. Note that T^* is also a Calderón–Zygmund operator, associated with the kernel k^* where: $k^*(x, y) = k(y, x)^*$.

So k^* satisfies the smoothness condition of parameter γ , which means by (7) that:

$$\|(T^*)_{k,s}\|_{\mathcal{B}(L_2(\mathcal{N}))} \lesssim 2^{-s\gamma}.$$

Consequently:

$$\|\Delta_{k+i} T_k\|_{\mathcal{B}(L_2(\mathcal{N}))} = \|T_k^* \Delta_{k+i}\|_{\mathcal{B}(L_2(\mathcal{N}))} = \|(T^*)_{k,i} \Delta_{k+i}\|_{\mathcal{B}(L_2(\mathcal{N}))} \lesssim 2^{-\gamma i},$$

where we used that $(T_k)^* = (T^*)_k$ and (6).

By orthogonality of the Δ_k , we have on one hand:

$$\|\Phi_i\|_{\mathcal{B}(L_2(\mathcal{N}))} \leq \sup_k \|\Delta_{k+i} T_k \Delta_{k+s}\|_{\mathcal{B}(L_2(\mathcal{N}))} \leq \sup_k \|\Delta_{k+i} T_k\|_{\mathcal{B}(L_2(\mathcal{N}))} \lesssim 2^{-\gamma i},$$

and on the other hand:

$$\|\Phi_i\|_{\mathcal{B}(L_2(\mathcal{N}))} \leq \sup_k \|\Delta_{k+i} T_k \Delta_{k+s}\|_{\mathcal{B}(L_2(\mathcal{N}))} \leq \sup_k \|T_k \Delta_{k+s}\|_{\mathcal{B}(L_2(\mathcal{N}))} \lesssim 2^{-\gamma s}.$$

By combining the two, $\|\Phi_i\|_{\mathcal{B}(L_2(\mathcal{N}))} \lesssim 2^{-\gamma \frac{s+i}{2}}$.

2.6. Estimate for Ψ_i

Lemma 2.7. *For all $x \in \mathbb{R}^n$, $k \in \mathbb{Z}$ and $i \in \mathbb{N}$, the following estimate holds:*

$$\int_{t \in Q_{x,k-i}} \int_{y \in Q_{x,k-i}^c} \mathbf{1}_{|t-y| > 2^{1-k}} \frac{1}{|t-y|^{n+\gamma}} dy dt \lesssim (1+i) 2^{(\gamma-n)(k-i)}.$$

Proof. Fix $x \in \mathbb{R}^n$, $k \in \mathbb{Z}$ and $i \in \mathbb{N}$. Denote $Q = Q_{x,k-i}$, c the centre of Q and:

$$X = \int_{t \in Q} \int_{y \in Q^c} \mathbf{1}_{|t-y| > 2^{1-k}} \frac{1}{|t-y|^{n+\gamma}} dy dt.$$

For every t in Q let δ_t be the distance from t to Q^c and notice that for any t :

$$\int_{Q^c} \mathbf{1}_{|t-y| > 2^{1-k}} \frac{1}{|t-y|^{n+\gamma}} dy \leq \int_{|y-t| > \delta_t} \mathbf{1}_{|t-y| > 2^{1-k}} \frac{1}{|t-y|^{n+\gamma}} dy =: f(\delta_t)$$

Indeed, the term on the right only depends on δ_t . For $r \leq 2^{-(k-i+1)}$:

$$\{t \in Q : \delta_t = r\} = \{t \in Q : |t-c| = 2^{-(k-i+1)} - r =: r'\}.$$

So by a “polar” change of coordinates:

$$X \leq \int_0^{2^{-(k-i+1)}} 2n(2r')^{n-1} f(r) dr \lesssim 2^{-(k-i)(n-1)} \int_0^{2^{-(k-i+1)}} f(r) dr.$$

By a direct computation:

$$f(r) \lesssim \begin{cases} r^{-\gamma} & \text{for all } r \\ 2^{\gamma(k-1)} & \text{if moreover } r \leq 2^{-(k-1)} \end{cases}$$

For $0 < \gamma < 1$ we only need the first estimate:

$$\begin{aligned} X &\lesssim 2^{-(k-i)(n-1)} \int_0^{2^{-(k-i+1)}} f(r) dr && \lesssim 2^{-(k-i)(n-1)} \int_0^{2^{-(k-i+1)}} r^{-\gamma} dr \\ &\lesssim 2^{-(k-i)(n-1)} 2^{-(k-i+1)(1-\gamma)} && \lesssim 2^{-n(k-i)} 2^{\gamma(k-i)} \end{aligned}$$

For $\gamma = 1$ we have to decompose the integral according to the distinction made above when estimating f :

$$\begin{aligned} \int_0^{2^{-(k-i+1)}} f(r)dr &= \int_0^{2^{-(k-1)}} f(r)dr + \int_{2^{-(k-1)}}^{2^{-(k-i+1)}} f(r)dr \\ &\lesssim 2^{-(k-1)}2^{\gamma(k-1)} + \int_{2^{-(k-1)}}^{2^{-(k-i+1)}} \frac{1}{r}dr \\ &\lesssim 1+i \end{aligned}$$

Therefore:

$$X \lesssim 2^{-(k-i)(n-1)}(1+i) = (1+i)2^{-n(k-i)}2^{\gamma(k-i)}. \quad \square$$

Proposition 2.8. *For all k in \mathbb{Z} and $i > 0$, we have the following estimate:*

$$\|\mathcal{E}_{k-i}S_{k,s}\|_{\mathcal{B}(L_2(\mathcal{N}))} \lesssim \sqrt{1+i}2^{-\gamma(s+i/2)}$$

Proof. For any function $g : \mathcal{E}_{k-i}S_{k,s}g(x) = \int_{Q_{x,k-i}} \frac{1}{V_{k-i}} \int_{\mathbb{R}^n} s_{k,s}(t,y)g(y)dydt$. So $\mathcal{E}_{k-i}S_{k,s}$ corresponds to the kernel:

$$E : (x,y) \mapsto \frac{1}{V_{k-i}} \int_{Q_{x,k-i}} s_{k,s}(t,y)dt = \frac{1}{V_{k-i}} \int_{Q_{x,k-i}^c} s_{k,s}(t,y)dt,$$

where we used the cancellation property on $S_{k,s}$. We are looking to estimate both integrals in order to apply Schur's lemma. Fix x :

$$\begin{aligned} \int_{\mathbb{R}^n} \|E(x,y)\|_{\widetilde{\mathcal{M}}} dy &= \int_{Q_{x,k-i}} \|E(x,y)\|_{\widetilde{\mathcal{M}}} dy + \int_{Q_{x,k-i}^c} \|E(x,y)\|_{\widetilde{\mathcal{M}}} dy \\ &= \int_{Q_{x,k-i}} \left\| \int_{Q_{x,k-i}^c} \frac{1}{V_{k-i}} s_{k,s}(t,y)dt \right\|_{\widetilde{\mathcal{M}}} dy \\ &\quad + \int_{Q_{x,k-i}^c} \left\| \int_{Q_{x,k-i}} \frac{1}{V_{k-i}} s_{k,s}(t,y)dt \right\|_{\widetilde{\mathcal{M}}} dy \\ &\lesssim \frac{1}{V_{k-i}} \int_{Q_{x,k-i}} \int_{Q_{x,k-i}^c} \mathbf{1}_{|t-y|>2^{1-k}} \frac{2^{-\gamma(k+s)}}{|t-y|^{n+\gamma}} dt dy \end{aligned}$$

Using [Lemma 2.7](#):

$$\begin{aligned} \int_{\mathbb{R}^n} \|E(x, y)\|_{\widetilde{\mathcal{M}}} dy &\lesssim \frac{1}{V_{k-i}} (1+i) 2^{-n(k-i)} 2^{\gamma(k-i)} 2^{-\gamma(k+s)} \\ &\lesssim (1+i) 2^{-\gamma(i+s)} \end{aligned}$$

The other estimate, with y fixed, is straightforward:

$$\begin{aligned} \int_{\mathbb{R}^n} \|E(x, y)\|_{\widetilde{\mathcal{M}}} dx &= \int_{\mathbb{R}^n} \left\| \frac{1}{V_{k-i}} \int_{Q_{x, k-i}} s_{k,s}(t, y) dt \right\|_{\widetilde{\mathcal{M}}} dx \\ &\leq \int_{\mathbb{R}^n} \frac{1}{V_{k-i}} \int_{Q_{x, k-i}} \|s_{k,s}(t, y)\|_{\widetilde{\mathcal{M}}} dt dx \\ &= \int_{\mathbb{R}^n} \|s_{k,s}(t, y)\|_{\widetilde{\mathcal{M}}} dt \\ &\lesssim 2^{-\gamma s} \end{aligned}$$

The proposition follows directly from the two previous computations and Schur's lemma. \square

We now have:

$$\|\Delta_{k-i} S_{k,s} \Delta_{k+s}\|_{\mathcal{B}(L_2(\mathcal{N}))} \leq \|\mathcal{E}_{k-i} S_{k,s}\|_{\mathcal{B}(L_2(\mathcal{N}))} \lesssim \sqrt{1+i} 2^{-\gamma \frac{s+i}{2}}.$$

We conclude once more thanks to the orthogonality between the $\Delta_{k-i} S_{k,s} \Delta_{k+s}$:

$$\|\Psi_i\|_{\mathcal{B}(L_2(\mathcal{N}))} \lesssim \sup_k \|\Delta_{k-i} S_{k,s} \Delta_{k+s}\|_{\mathcal{B}(L_2(\mathcal{N}))} \lesssim \sqrt{1+i} 2^{-\gamma \frac{s+i}{2}}$$

This concludes the proof of the pseudo-localisation theorem.

3. Proof of the main theorem

3.1. The good and the bad functions

The idea of the following decomposition comes from the classical proof of the weak type inequality for singular integrals. It has been noticed that the commutative decomposition can be expressed in terms of martingales which is well suited for a translation in the noncommutative setting. However, even with this idea, the construction is not immediate and the estimates are more difficult to obtain due to the appearance of new “off-diagonal” terms.

Fix $t > 0$ and $f \in L_1(\mathcal{N})$. We will suppose that f is positive to avoid unnecessary computations. This is possible because f can always be written as:

$$f = f_1 - f_2 + if_3 - if_4$$

with $\|f_k\|_1 \leq \|f\|_1$ for $k = 1, 2, 3, 4$. Suppose also that $\{x \in \mathbb{R}^n : f(x) \neq 0\}$ is bounded (in other words, f , considered as a function, has a compact support), and that f is in $L_2(\mathcal{N}) \cap \mathcal{N}$. We can make these assumptions since the functions satisfying them form a dense subspace \mathcal{S} of $L_1(\mathcal{N})$. Once the theorem is proven, T can be defined on $L_1(\mathcal{N})$ as the only bounded extension of its restriction to \mathcal{S} .

Denote by $(f_n)_{n \in \mathbb{Z}}$ the martingale associated with f and the filtration $(\mathcal{N}_n)_{n \in \mathbb{Z}}$. The main tools to decompose f into a good and a bad part are Cuculescu's projections:

Theorem 3.1 (Cuculescu). *Let $x = (x_n)$ be a bounded positive L_1 -martingale and $t \geq 0$. Then there exists a decreasing sequence (q_n) of projections in \mathcal{N} such that for every $n \geq 1$:*

1. $q_n \in \mathcal{N}_n$.
2. q_n commutes with $q_{n-1}x_nq_{n-1}$.
3. $q_nx_nq_n \leq t$.
4. Moreover, if $q = \bigwedge q_n$ then:

$$qx_nq \leq t \text{ for } n \geq 1 \text{ and } \tau(q^\perp) \leq \frac{\|x\|_1}{t}.$$

The boundedness hypothesis on f and its support imply that there exists n_0 such that for all $n \leq n_0$, $f_n \leq t$. By a change of variables, we do not lose generality assuming that $n_0 = 0$. From now on, let q_n denote the projections given by Cuculescu's theorem, associated with t and $(f_n)_{n \geq 0}$. To complete this definition let $q_n = 1$ for all $n \leq 0$. Notice that for all $n \in \mathbb{Z}$, $q_nf_nq_n \leq t$.

Define:

$$\forall n \in \mathbb{Z}, p_n = q_{n-1} - q_n \text{ and } p_\infty = q.$$

Let $\overline{\mathbb{Z}} = \mathbb{Z} \cup \{\infty\}$. By definition, $p_n \in \mathcal{N}_n$ and:

$$\sum_{n \in \overline{\mathbb{Z}}} p_n = 1.$$

Which allows us to define the good and bad parts as follows:

$$g = \sum_{i \in \overline{\mathbb{Z}}} \sum_{j \in \overline{\mathbb{Z}}} p_i f_{i \vee j} p_j \text{ and } b = \sum_{i \in \overline{\mathbb{Z}}} \sum_{j \in \overline{\mathbb{Z}}} p_i (f - f_{i \vee j}) p_j.$$

By properties of the distribution function (see [3]):

$$\lambda_t(Tf) \lesssim \lambda_{t/2}(Tg) + \lambda_{t/2}(Tb)$$

So it suffices to prove estimates of the form:

$$\lambda_t(Tx) \lesssim \frac{\|f\|_1}{t}$$

for both $x = g$ and $x = b$. This is the purpose of the next sections.

Remark 3.2. The same formula for b and g works in the commutative case except that only the diagonal terms are non zero. This explains why the commutative proof can only be repeated for the diagonal terms and “shifted” diagonals. What makes this decomposition work is that, due to the pseudo-localisation lemma, the estimates for “shifted” diagonals get exponentially better as the shift increases.

We will use the following notation: for all k and $Q \in \mathcal{Q}_k$, $p_Q := p_k(x)$ for any $x \in Q$. We will also need two lemmas which are directly deduced from the construction.

Lemma 3.3. *For all $k \in \mathbb{Z}$, we have: $f_{k+1} \leq 2^n f_k$.*

Proof. This is straightforward from the definition of f_k and positivity of f . Let $x \in \mathbb{R}^n$:

$$2^n f_k(x) = \frac{2^n}{V_k} \int_{Q_{x,k}} f(t) dt \geq \frac{2^n}{V_k} \int_{Q_{x,k+1}} f(t) dt = f_{k+1}(x). \quad \square$$

Lemma 3.4. *Let d be an odd positive integer. Define:*

$$\zeta_d = \left(\bigvee_{Q \in \mathcal{Q}} p_Q \mathbf{1}_{dQ} \right)^\perp.$$

Then:

1. $\tau(\zeta_d^\perp) \leq d^n \frac{\|f\|_1}{t}$
2. *For all cubes $Q \in \mathcal{Q}$, we have the following cancellation property:*

$$x \in dQ \Rightarrow \zeta_d(x) p_Q = p_Q \zeta_d(x) = 0.$$

Proof. The first estimate is a consequence of Cuculescu’s inequality:

$$\tau(\zeta_d^\perp) \leq \sum_{Q \in \mathcal{Q}} d^n \tau(p_Q \mathbf{1}_Q) = d^n \sum_{k=1}^{\infty} \tau(p_k) = d^n \tau(q^\perp) \leq d^n \frac{\|f\|_1}{t}.$$

Let $Q \in \mathcal{Q}$ and $x \in dQ$. By construction, $\zeta_d^\perp(x) \geq p_Q$ so $\zeta_d(x) \leq p_Q^\perp$. This concludes the proof. \square

From now on, we fix $d = 5$ and denote ζ_d by ζ .

Remark 3.5. This projection ζ should be thought of as a dilation of the support of the bad function. It already plays a crucial role in the commutative setting.

3.2. Estimate for the bad function

The strategy of proof is to write:

$$Tb = \zeta Tb\zeta + (1 - \zeta)Tb\zeta + \zeta Tb(1 - \zeta) + (1 - \zeta)Tb(1 - \zeta).$$

Therefore, Lemma 3.4 and Tchebychev's inequality give:

$$\lambda_t(Tb) \lesssim \tau(1 - \zeta) + \lambda_t(\zeta Tb\zeta) \lesssim \frac{\|f\|_1}{t} + \frac{\|\zeta Tb\zeta\|_1}{t}.$$

The estimate for the bad function is now reduced to the following proposition:

Proposition 3.6. *We have the estimate: $\|\zeta Tb\zeta\|_1 \lesssim \|f\|_1$.*

The proof will require three intermediate lemmas.

Define, for all $i, j \in \mathbb{Z}$: $b_{i,j} = p_i(f - f_{i \vee j})p_j$.

Lemma 3.7. *For all $s \in \mathbb{Z}$: $\sum_{i-j=s} \|b_{i,j}\|_1 \lesssim \|f\|_1$*

Proof. Let $i, j \in \mathbb{Z}$:

$$\|b_{i,j}\|_1 \leq \|p_i f p_j\|_1 + \|p_i f_{i \vee j} p_j\|_1$$

By Holder's inequality:

$$\begin{aligned} \|b_{i,j}\|_1 &\leq \|f^{1/2} p_i\|_2 \|f^{1/2} p_j\|_2 + \|f_{i \vee j}^{1/2} p_i\|_2 \|f_{i \vee j}^{1/2} p_j\|_2 \\ &\leq \frac{1}{2} (\|p_i f p_i\|_1 + \|p_j f p_j\|_1 + \|p_i f_{i \vee j} p_i\|_1 + \|p_j f_{i \vee j} p_j\|_1) \\ &\leq \tau(p_i f) + \tau(p_j f) \end{aligned}$$

Consequently, for all $s \in \mathbb{Z}$:

$$\sum_{i-j=s} \|b_{i,j}\|_1 \leq 2 \sum_{i \in \mathbb{Z}} \tau(p_i f) \lesssim \|f\|_1 \quad \square$$

Lemma 3.8. *The following cancellation properties hold:*

- for all $i, j \in \mathbb{Z}$ and $Q \in \mathcal{Q}_{i \vee j}$: $\int_Q b_{i,j} = 0$;
- for all $x, y \in \mathbb{R}^n$ such that $y \in 5Q_{x, i \wedge j}$: $\zeta(x) b_{i,j}(y) \zeta(x) = 0$.

Proof. The first point is straightforward. Since $Q \in \mathcal{Q}_{i \vee j}$: $\int_Q b_{i,j} = \int_Q \mathcal{E}_{i \vee j}(b_{i,j}) = 0$.

The second one is a consequence of the construction of ζ expressed in [Property 3.4](#). Indeed, for all $x, y \in \mathbb{R}^n$ and $k \in \mathbb{Z}$ such that $y \in 5Q_{x,k}$ we know that $\zeta(x)p_k(y) = p_k(y)\zeta(x) = 0$. Recall that $b_{i,j} = p_i(f - f_{i \vee j})p_j$, it is now clear that $\zeta(x)b_{i,j}(y)\zeta(x) = 0$ for $y \in 5Q_{x,k}$ and $k = i, j$ which concludes the proof of the lemma. Note that $Q_{x,k} \subset Q_{x,k'}$ for $k \geq k'$, so we do not lose anything by taking $k = i \wedge j$. \square

The following lemma is the core of the bad function estimate, it relies on a computation which allows us to make use of the smoothness condition.

Lemma 3.9. For all $i, j \in \mathbb{Z}$: $\|\zeta T b_{i,j} \zeta\|_1 \lesssim 2^{-|i-j|\gamma} \|b_{i,j}\|_1$.

Proof. Note that $b_{i,j}$ is in $L_2(\mathcal{N})$. Fix $i, j \in \mathbb{Z}$ and $x \in \mathbb{R}^n$, recall that $k(x, y)$ commutes with $\zeta(x)$ since ζ takes values in \mathcal{M} and k in \mathcal{M}' :

$$\begin{aligned} \zeta(x) T b_{i,j}(x) \zeta(x) &= \int_{y \in \mathbb{R}^n} k(x, y) \zeta(x) b_{i,j}(y) \zeta(x) dy \\ &= \int_{y \in 5Q_{x, i \wedge j}^c} (k(x, y) - k(x, c_{y, i \vee j})) \zeta(x) b_{i,j}(y) \zeta(x) dy \end{aligned}$$

where we used both cancellation properties of $b_{i,j}$ ([Lemma 3.8](#)) the first one to make the term $k(x, c_{y, i \vee j})$ appear and the second one to reduce the domain of integration. Therefore, the smoothness condition ([Definition 1.3](#)) applies and gives:

$$\begin{aligned} \|\zeta(x) T b_{i,j}(x) \zeta(x)\|_1 &\leq \int_{y \in 5Q_{x, i \wedge j}^c} \|k(x, y) - k(x, c_{y, i \vee j})\|_{\widetilde{\mathcal{M}}} \|b_{i,j}(y)\|_1 dy \\ &\leq \int_{y \in \mathbb{R}^n} \mathbf{1}_{y \notin 5Q_{x, i \wedge j}} \frac{2^{-\gamma(i \vee j)}}{|x - y|^{n+\gamma}} \|b_{i,j}(y)\|_1 dy \end{aligned}$$

It follows that:

$$\begin{aligned} \|\zeta T b_{i,j} \zeta\|_1 &= \int_{x \in \mathbb{R}^n} \|\zeta(x) T b_{i,j}(x) \zeta(x)\|_1 dx \\ &\leq \int_{y \in \mathbb{R}^n} \int_{x \in \mathbb{R}^n} \mathbf{1}_{x \notin 5Q_{y, i \wedge j}} \frac{2^{-\gamma(i \vee j)}}{|x - y|^{n+\gamma}} \|b_{i,j}(y)\|_1 dx dy \\ &\lesssim \int_{y \in \mathbb{R}^n} 2^{-\gamma(i \vee j - i \wedge j)} \|b_{i,j}(y)\|_1 dy \\ &\lesssim 2^{-\gamma|i-j|} \|b_{i,j}\|_1 \quad \square \end{aligned}$$

Proof of Property 3.6. The only thing left to do is to glue the pieces together thanks to Lemmas 3.7 and 3.9:

$$\begin{aligned}\|\zeta T b \zeta\|_1 &\leq \sum_{i,j \in \mathbb{Z}} \|\zeta T b_{i,j} \zeta\|_1 \leq \sum_{s \in \mathbb{Z}} \sum_{i-j=s} \|\zeta T b_{i,j} \zeta\|_1 \\ &\leq \sum_{s \in \mathbb{Z}} 2^{-\gamma|s|} \sum_{i-j=s} \|b_{i,j}\|_1 \lesssim \sum_{s \in \mathbb{Z}} 2^{-\gamma|s|} \|f\|_1 \lesssim \|f\|_1. \quad \square\end{aligned}$$

3.3. Estimate for the good function

This one is more involved and requires the L_2 -pseudo-localisation theorem. The same trick as for the bad function allows us to write:

$$\lambda_t(Tg) \lesssim \tau(1 - \zeta) + \lambda_t(\zeta T g \zeta) \lesssim \frac{\|f\|_1}{t} + \frac{\|\zeta T g \zeta\|_2^2}{t^2}.$$

Define the diagonal, left and right parts of g as follows:

$$\begin{aligned}g^{(d)} &= \sum_{i \in \mathbb{Z}} p_i f_i p_i, g^{(l)} = \left(\sum_{i < j \in \mathbb{Z}} p_i f_j p_j \right) + q^\perp f q \text{ and} \\ g^{(r)} &= \left(\sum_{i < j \in \mathbb{Z}} p_j f_j p_i \right) + q f q^\perp.\end{aligned}$$

Note that the estimate for the good function can easily be deduced from the following property:

Proposition 3.10. *The following estimates hold:*

$$\|Tg^{(d)}\|_2^2 \lesssim t\|f\|_1, \quad \|\zeta Tg^{(l)}\|_2^2 \lesssim t\|f\|_1 \text{ and } \|Tg^{(r)}\zeta\|_2^2 \lesssim t\|f\|_1$$

Proof of the $g^{(d)}$ estimate. Since T is bounded on $L_2(\mathcal{N})$ it suffices to prove that $\|g^{(d)}\|_2^2 \lesssim t\|f\|_1$.

Notice that $g^{(d)}$ is positive.

$$\|g^{(d)}\|_1 = \sum_{i \in \mathbb{Z}} \tau(p_i f_i p_i) = \sum_{i \in \mathbb{Z}} \tau(p_i f) = \tau\left(\sum_{i \in \mathbb{Z}} p_i f\right) = \|f\|_1.$$

By orthogonality, we have $\|g^{(d)}\|_\infty = \sup_{k \in \mathbb{Z}} \|p_k f_k p_k\| \leq 2^n t$. Indeed, for $k < \infty$, we have, by Lemma 3.3:

$$p_k f_k p_k \leq 2^n p_k f_{k-1} p_k \leq 2^n q_{k-1} f_{k-1} q_{k-1} \leq 2^n t$$

For $k = \infty$, reasoning in $L_2(\mathcal{N})$: $t - qfq = \lim_{k \rightarrow \infty} t - qf_k q \geq 0$ since $L_2(\mathcal{N})^+$ is closed. So $\|g_d\|_\infty \leq 2^nt$.

We conclude by Hölder's inequality:

$$\|g_d\|_2^2 \leq \|g_d\|_1 \|g_d\|_\infty \leq 2^nt \|f\|_1. \quad \square$$

Proof of the $g^{(l)}$ estimate. This will conclude the proof of [Proposition 3.10](#) since the argument for $g^{(r)}$ is similar. \square

Lemma 3.11. *We have the following expression for $g^{(l)}$:*

$$g^{(l)} = \sum_{k=1}^{\infty} \sum_{s=1}^{\infty} p_k df_{k+s} q_{k+s-1} =: \sum_{k=1}^{\infty} \sum_{s=1}^{\infty} g_{s,k}^{(l)}.$$

Proof. This is obtained by an Abel's transform. First, fix j_0 :

$$\begin{aligned} \sum_{i < j \leq j_0} p_i f_j p_j &= \sum_{i < j \leq j_0} p_i f_j (q_{j-1} - q_j) = \sum_{i \leq j < j_0} p_i df_{j+1} q_j - \sum_{i < j_0} p_i f_{j_0} q_{j_0} \\ &= \sum_{i \leq j < j_0} p_i df_{j+1} q_j - q_{j_0-1}^\perp f_{j_0} q_{j_0} \end{aligned}$$

Letting j_0 go to infinity, we obtain:

$$\sum_{i < j \in \mathbb{Z}} p_i f_j p_j = \left(\sum_{i \leq j \in \mathbb{Z}} p_i df_{j+1} q_j \right) - q^\perp f q,$$

which is exactly what we needed. \square

Proposition 3.12. *Define, for all $s \geq 1$:*

$$g_s^{(l)} = \sum_{k=1}^{\infty} p_k df_{k+s} q_{k+s-1}$$

Then: $\|\zeta T g_s^{(l)}\|_2^2 \lesssim 2^{-\gamma s t} \|f\|_1$.

This is, as for the bad function, the core of the proof and will require some work.

Lemma 3.13. *We have the following properties:*

1. $\Delta_{k+s}(g_s^{(l)}) = g_{s,k}^{(l)}$.
2. Fix s , the $g_{s,k}^{(l)}$ are orthogonal in $L_2(\mathcal{N})$.
3. $\|g_s^{(l)}\|_2^2 \lesssim t \|f\|_1$.

Proof. 1 is straightforward and 2 follows directly from 1 since martingale differences are always orthogonal. Let us prove 3.

$$\begin{aligned}\|g_{s,k}^{(l)}\|_2^2 &\leq 2(\|p_k f_{k+s} q_{k+s-1}\|_2^2 + \|p_k f_{k+s-1} q_{k+s-1}\|_2^2) \\ &\leq 2\tau(p_k f_{k+s} q_{k+s-1} f_{k+s} p_k) + 2\tau(p_k f_{k+s-1} q_{k+s-1} f_{k+s-1} p_k)\end{aligned}$$

By Cuculescu's theorem and recalling that $f_i \leq 2^n f_{i-1}$ from [Lemma 3.3](#):

$$\begin{aligned}\|f_{k+s}^{1/2} q_{k+s-1} f_{k+s}^{1/2}\|_\infty &= \|q_{k+s-1} f_{k+s} q_{k+s-1}\|_\infty \lesssim t \\ \|f_{k+s}^{1/2} q_{k+s-1} f_{k+s-1}^{1/2}\|_\infty &= \|q_{k+s-1} f_{k+s-1} q_{k+s-1}\|_\infty \leq t\end{aligned}$$

Therefore, for all $s > 0$:

$$\|g_s^{(l)}\|_2^2 = \sum_{k>0} \|g_{k,s}^{(l)}\|_2^2 \lesssim \sum_{k>0} t\tau(p_k f) = t\|f\|_1. \quad \square$$

Lemma 3.14. *We have the following estimate:*

$$\|\zeta T g_s^{(l)}\|_2 \lesssim 2^{-\gamma s/2} \|g_s^{(l)}\|_2$$

Proof. This is where we apply pseudo-localisation i.e. [Theorem 2.1](#) to $g_s^{(l)}$. Using the notations introduced for this theorem, we can take $A_k = p_k$ since $p_k^\perp g_{s,k}^{(l)} = 0$. Then:

$$A_{f,s} = \bigvee_{k>0} 5p_k = \zeta^\perp.$$

The theorem gives:

$$\|A_{f,s}^\perp T g_s^{(l)}\|_2 \lesssim 2^{-\gamma s/2} \|g_s^{(l)}\|_2$$

which is exactly the expected estimate. \square

The [Proposition 3.12](#) is clear from the two previous lemmas.

It follows that:

$$\begin{aligned}\|\zeta T g^{(l)}\|_2^2 &\leq \left(\sum_{s=1}^{\infty} \|\zeta T g_s^{(l)}\|_2\right)^2 \leq \left(\sum_{s=1}^{\infty} 2^{-\frac{\gamma s}{2}} \|g_s^{(l)}\|_2\right)^2 \\ &\lesssim \left(\sum_{s=1}^{\infty} 2^{-\frac{\gamma s}{2}} \sqrt{t\|f\|_1}\right)^2 \lesssim t\|f\|_1\end{aligned}$$

which is the expected estimate for $g^{(l)}$ and concludes the proof of the main theorem.

4. Application to Hilbert-valued kernels

Let (\mathcal{M}, τ) be a noncommutative measure space, $\mathcal{N} = L_\infty(\mathbb{R}) \overline{\otimes} \mathcal{M}$. Let T be a Calderón–Zygmund operator associated to a kernel k taking values in ℓ^2 . For all $1 \leq p \leq \infty$, we have a multiplication from $\ell^2 \times L_p(\mathcal{M})$ to $L_p(\mathcal{M})^\mathbb{N}$: for all $h = (h_i)_{i \in \mathbb{N}} \in \ell^2$ and $x \in \mathcal{M}$, define $hx = (h_i x)_{i \in \mathbb{N}}$.

There are different natural norms on $L_p(\mathcal{M})^\mathbb{N}$ that make this multiplication continuous. We will be interested in (see [13]):

$$\|x\|_{C_p(\mathcal{M})} = \left\| \left(\sum_{k=0}^{\infty} x_k^* x_k \right)^{1/2} \right\|_p, \quad \|x\|_{R_p(\mathcal{M})} = \left\| \left(\sum_{k=0}^{\infty} x_k x_k^* \right)^{1/2} \right\|_p$$

and

$$\|x\|_{RC_p(\mathcal{M})} = \begin{cases} \inf \{ \|y\|_{C_p(\mathcal{M})} + \|z\|_{R_p(\mathcal{M})} : y + z = x \} & \text{if } p < 2 \\ \max(\|x\|_{C_p(\mathcal{M})}, \|x\|_{R_p(\mathcal{M})}) & \text{if } p \geq 2. \end{cases}$$

Denote by $C_p(\mathcal{M})$, $R_p(\mathcal{M})$ and $RC_p(\mathcal{M})$ the associated subspaces of $L_p(\mathcal{M})^\mathbb{N}$. Likewise, define $C_p(\mathcal{N})$, $R_p(\mathcal{N})$ and $RC_p(\mathcal{N})$. Note that $C_2(\mathcal{M}) = R_2(\mathcal{M}) = RC_2(\mathcal{M}) = \ell^2(L_2(\mathcal{M}))$.

Thanks to the multiplication we defined, T can be seen formally as an operator from $L_p(\mathcal{N})$ to $L_p(\mathcal{N})^\mathbb{N}$ by the usual expression:

$$Tf(x) = \int_{\mathbb{R}} k(x, y) f(y) dy.$$

We will show in this section that T is bounded from $L_p(\mathcal{N})$ to $RC_p(\mathcal{N})$.

To apply the theorem, we have to include \mathcal{M} and ℓ^2 in a von Neumann algebra such that their images commute. Let \mathbb{F}_∞ be the free group with countably many generators $(g_n)_{n \in \mathbb{N}}$, λ its left regular representation and $C(\mathbb{F}_\infty)$ its associated von Neumann algebra. Define $\widetilde{\mathcal{M}} = C(\mathbb{F}_\infty) \otimes \mathcal{M}$, $\widetilde{\mathcal{N}} = L_\infty(\mathbb{R}) \otimes \widetilde{\mathcal{M}}$ and the inclusion maps $i_1 : \mathcal{M} \rightarrow \widetilde{\mathcal{M}}$ and $i_2 : \ell^2 \rightarrow \widetilde{\mathcal{M}}$ by:

$$i_1(x) = 1 \otimes x \text{ and } i_2(h) = \sum_{k=0}^{\infty} h_k \lambda(g_k) \otimes 1.$$

From now on \mathcal{M} will also designate $1 \otimes \mathcal{M}$ and can be considered as a von Neumann subalgebra of $\widetilde{\mathcal{M}}$ since $\tau_{\widetilde{\mathcal{M}}}$ coincides with $\tau_{\mathcal{M}}$ on \mathcal{M} . Note that the image of i_2 is in $\mathcal{M}' \cap \widetilde{\mathcal{M}}$. Let $\tilde{k} : (x, y) \mapsto i_2(k(x, y))$ be the kernel of a Calderón–Zygmund operator $\tilde{T} : L_p(\mathcal{N}) \rightarrow L_p(\widetilde{\mathcal{N}})$. The image of i_2 is in \mathcal{M}' . The following lemma (Khinchine's inequalities for the free group) is the crucial point of the construction (see [11], [10]).

Lemma 4.1. Let $a = (a_n)_{n \in \mathbb{N}}$ be a sequence in $L_p(\mathcal{M})^{\mathbb{N}}$, then for all $1 \leq p \leq \infty$:

$$\|a\|_{RC_p(\mathcal{M})} \approx \left\| \sum_{n=0}^{\infty} \lambda(g_n) \otimes a_n \right\|_{L_p(\widetilde{\mathcal{M}})}$$

Corollary 4.2. Let T be a Calderón–Zygmund operator associated with a kernel k taking values in ℓ^2 and \widetilde{T} as defined above, then for all $1 \leq p \leq \infty$ and all $f \in L_p(\mathcal{N})$:

$$\|Tf\|_{RC_p(\mathcal{N})} \approx \|\widetilde{T}f\|_{L_p(\widetilde{\mathcal{N}})}.$$

Proof. Notice that if $Tf = (T_i f)_{i \in \mathbb{N}}$ in $RC_p(\mathcal{N})$ then $\widetilde{T}f = \sum_{i=0}^{\infty} \lambda(g_n) \otimes T_i f$ in $L_p(\widetilde{\mathcal{N}})$ and apply the previous lemma. \square

Proposition 4.3. Let T be a Calderón–Zygmund operator associated to a kernel k taking values in ℓ^2 . Suppose furthermore that T is bounded from $L_2(\mathcal{N})$ to $RC_2(\mathcal{N})$ and that k satisfies the smoothness and size conditions then the operator \widetilde{T} defined above is bounded from $L_1(\mathcal{N})$ to $L_{1,\infty}(\widetilde{\mathcal{N}})$.

Proof. We only have to check that Theorem 1.5 can be applied to \widetilde{T} . Khintchine’s inequality for $p = \infty$ imply that for all $h \in \ell^2$, $\|h\| \approx \|i_2(h)\|_{\widetilde{\mathcal{M}}}$ so \widetilde{k} verifies the size and smoothness conditions. Furthermore, Corollary 4.2 applied to $p = 2$ gives the boundedness condition on L_2 . So all the hypothesis are verified and \widetilde{T} is bounded from $L_1(\mathcal{N})$ to $L_{1,\infty}(\widetilde{\mathcal{N}})$. \square

Corollary 4.4. Let T be a Calderón–Zygmund operator associated to a kernel k taking values in ℓ^2 . Suppose furthermore that T is bounded from $L_2(\mathcal{N})$ to $RC_2(\mathcal{N})$ and that k satisfies the smoothness and size condition. Then T is bounded from $L_p(\mathcal{N})$ to $RC_p(\mathcal{N})$ for all $1 < p \leq 2$.

Proof. From the previous property, it is clear that \widetilde{T} is bounded from $L_p(\mathcal{N})$ to $L_p(\widetilde{\mathcal{N}})$ by real interpolation. Which is enough to conclude thanks to Corollary 4.2. \square

Remark 4.5. T is not bounded from $L_p(\mathcal{N})$ to $C_p(\mathcal{N})$, for $1 < p < 2$. It is necessary to consider $RC_p(\mathcal{N})$.

Proof. We will construct a counter-example thanks to the Littlewood–Paley kernel (the same general idea can again be found in [9]).

Let ψ be a C^∞ function supported in $(1, 2)$, bounded by 1 and constant equal to one on $(5/4, 7/4)$. For any $i \in \mathbb{N}^*$, let $\phi_i : t \mapsto \psi(it)$. Consider the kernel $k : \mathbb{R}^2 \rightarrow \ell^2$ defined by $k_i(x, y) = \widehat{\phi_i}(x - y)$ where $\widehat{\phi_i}$ denotes the Fourier transform of ϕ_i . It is standard that k verifies the smoothness estimate for $\gamma = 1$ and the size condition.

The Calderón–Zygmund operator T associated to k is bounded on L_2 by Plancherel's theorem.

Let $\mathcal{M} = \mathcal{B}(\ell^2)$. As always, let $\mathcal{N} = L_\infty(\mathbb{R}) \otimes \mathcal{M}$. For all $p \geq 1$, T can be seen as an operator from $L_p(\mathcal{N})$ to $C_p(\mathcal{N})$. For all $k > 0$, let g_k be the Fourier transform of $\mathbf{1}_{(\frac{5}{4}2^{k-1}, \frac{5}{4}2^{k-1}+1/2)}$. Notice that $\widehat{\phi_i} * g_k = \delta$. Fix $m > 0$ an integer. Let $f_m = \sum_{k=1}^m g_k \otimes e_{1,k}$, then $Tf_m = (g_k \otimes e_{1,k})_{e_k \in \mathbb{N}^*}$. It results that $\|f_m\|_{L_p(\mathcal{N})} = m^{1/2}\|g_1\|_p$ and $\|Tf_m\|_{C_p(\mathcal{N})} = m^{1/p}\|g_1\|_p$. So T is not bounded for $p < 2$. \square

5. L_p -pseudo-localisation

In this section, we will recover in this context the main result from [5] which is an L_p -version of pseudo-localisation. The proof heavily relies on estimates and definitions introduced in section 2. Precisely, we will show the following theorem, using notations from Theorem 2.1.

Theorem 5.1. *Let T be a Calderón–Zygmund operator associated with a kernel k with Lipschitz parameter γ , verifying the size condition, taking values in $\mathcal{M}' \cap \widetilde{\mathcal{M}}$ and bounded from $L_2(\mathcal{N})$ to $L_2(\widetilde{\mathcal{N}})$. Then for all $1 < p < \infty$, $s \in \mathbb{N}$ and $f \in L_p(\mathcal{N})$, there exists $\theta_p > 0$ such that:*

$$\|A_{f,s}^\perp(Tf)\|_p \lesssim 2^{-\gamma s \theta_p} \|f\|_p \text{ and } \|(Tf)B_{f,s}^\perp\|_p \lesssim 2^{-\gamma s \theta_p} \|f\|_p,$$

where the implied constant depends on p .

We will use the decomposition we obtained in the L_2 -case in (9):

$$A_{f,s}^\perp Tf = A_{f,s}^\perp \left(\sum_{i=0}^{\infty} \Phi_i f + \sum_{i=1}^{\infty} \Psi_i f \right).$$

The crucial result, proved in the next sections, is the following:

Proposition 5.2. *For all $p \in (1, \infty)$ the sequences of operators $(\Phi_i)_{i \in \mathbb{N}}$ and $(\Psi_i)_{i \in \mathbb{N}}$ are uniformly bounded on $L_p(\mathcal{N})$.*

Proof of the theorem. Fix $p \in (1, \infty)$. By complex interpolation, there exist $q \in (1, \infty)$ and $\theta_p > 0$ such that $\|\Phi_i\|_{\mathcal{B}(L_p(\mathcal{N}))} \leq \|\Phi_i\|_{\mathcal{B}(L_2(\mathcal{N}))}^{2\theta_p} \|\Phi_i\|_{\mathcal{B}(L_q(\mathcal{N}))}^{1-2\theta_p}$.

Now, it suffices to plug in the previous proposition and Proposition 2.6 to get the expected estimate. We obtain: $\|\Phi_i\|_{\mathcal{B}(L_p(\mathcal{N}))} \lesssim 2^{-\theta_p(i+s)}$. A similar estimate is true for Ψ_i . This is enough to conclude thanks to the decomposition (9) mentioned above. \square

5.1. L_p -boundedness of Φ_i

Lemma 5.3. *Consider a Calderón–Zygmund operator T with kernel k , and an integer m . Then, for any $p \in (1, \infty)$, the kernel $k' : (x, y) \mapsto \mathbf{1}_{x \in 5Q_{y,m}} k(x, y)$ defines a bounded operator T' on $L_p(\mathcal{N})$.*

Proof. We introduce a sequence of Rademacher variables indexed by the set of dyadic cubes of size 2^{-m} to express the kernel k' in a suitable way. Let Ω be the probability space $\{-1, 1\}^{\mathcal{Q}_m}$ where \mathcal{Q}_m is the set of dyadic cubes of size 2^{-m} and $\tilde{\mathcal{N}} = \mathcal{N} \otimes L_\infty(\Omega)$. Consider \mathcal{N} as a subalgebra of $\tilde{\mathcal{N}}$. We will take a functional approach and work in $\tilde{\mathcal{N}}$ from now on. Denote by ε_Q the Q -coordinate in Ω . Define $g_m(x, \omega) = \sum_{Q \in \mathcal{Q}_m} \mathbf{1}_Q(x) \varepsilon_Q(\omega)$, $l_m(x, \omega) = \sum_{Q \in \mathcal{Q}_m} \mathbf{1}_{5Q}(x) \varepsilon_Q(\omega)$. Let G_m (resp. L_m) be the multiplication by g_m (resp. l_m) and \mathcal{E} be the conditional expectation onto $L_p(\mathcal{N})$. Note that $\|G_m\|_{\mathcal{B}(L_p(\mathcal{N}))} = \|g_m\|_\infty = 1$ and $\|L_m\|_{\mathcal{B}(L_p(\mathcal{N}))} = \|l_m\|_\infty = 5^n$.

It is easily checked that $T' = \mathcal{E} L_m T G_m$ since

$$k'(x, y) = \int_{\Omega} l_m(x, \omega) k(x, y) g_m(y, \omega) d\omega.$$

So T' can be extended to a bounded operator on $L_p(\mathcal{N})$. \square

Proposition 5.4. *Let $p \in (1, \infty)$. The operators Φ_i are uniformly bounded on $L_p(\mathcal{N})$.*

Proof. We identify $L_p(\mathcal{N})$ quasi-isometrically with a subspace of $RC_p(\tilde{\mathcal{N}})$, thanks to Burkholder–Gundy inequality (theorem 2.1 in [12]): $\|f\|_p \approx \|df\|_{RC_p(\tilde{\mathcal{N}})}$. Recall that $\Phi_i = \sum_{k \in \mathbb{Z}} \Delta_{k-s+i} T_{k-s} \Delta_k = \sum_{k \in \mathbb{Z}} \Delta_{k-s+i} (T - \mathcal{E} L_{k-s} T G_{k-s}) \Delta_k$, where the L_k and G_k are given by the lemma above. We can write $\Phi_i = \Delta T - \Delta \mathcal{E} L T G$ where $Gx = (G_{k-s} x_k)_{k \in \mathbb{Z}}$, $Lx = (L_{k-s} x_k)_{k \in \mathbb{Z}}$, $\Delta x = (\Delta_{k-s+i} x_k)_{k \in \mathbb{Z}}$ and $Tx = (Tx_k)_{k \in \mathbb{Z}}$. L and G are still bounded multiplications by unconditionality of $RC_p(\tilde{\mathcal{N}})$. T is completely bounded on $RC_p(\tilde{\mathcal{N}})$ as a diagonal Calderón–Zygmund operator, \mathcal{E} as a conditional expectation from $RC_p(\tilde{\mathcal{N}})$ to $RC_p(\mathcal{N})$ and Δ by Stein’s inequality (Theorem 2.3 in [12]):

$$\|(\mathcal{E}_n(x_n))\|_{L_p(\tilde{\mathcal{N}}, \ell_2^n)} \leq \|(x_n)\|_{L_p(\tilde{\mathcal{N}}, \ell_2^n)} \quad \text{and} \quad \|(\mathcal{E}_n(x_n))\|_{L_p(\tilde{\mathcal{N}}, \ell_2^n)} \leq \|(x_n)\|_{L_p(\tilde{\mathcal{N}}, \ell_2^n)}.$$

Hence, the Φ_i are uniformly bounded. \square

5.2. L_p -boundedness of Ψ_i

Recall that,

$$\Psi_i = \sum_{k \in \mathbb{Z}} \Delta_{k-i} S_{k,s} \Delta_{k+s} = \sum_{k \in \mathbb{Z}} \Delta_{k-i} (T_{k,s} + R_{k,s}) \Delta_{k+s} = A_i + B_i$$

where $A_i = \sum_{k \in \mathbb{Z}} \Delta_{k-i} T_{k,s} \Delta_{k+s} = \sum_{k \in \mathbb{Z}} \Delta_{k-i} T_k \Delta_{k+s}$ (by 6) and $B_i = \sum_{k \in \mathbb{Z}} \Delta_{k-i} R_{k,s} \Delta_{k+s}$. The previous section tells us that A_i is bounded from $L_p(\mathcal{N})$ to $L_p(\mathcal{N})$, with a control on the norm independent from i . So what is left to prove is the following proposition.

Proposition 5.5. *Let $p \in (1, \infty)$, then B_i is bounded from $L_p(\mathcal{N})$ to $L_p(\mathcal{N})$, uniformly in i .*

Proof. Using notations from the previous section, we will again consider B_i as a partial operator from $RC_p(\tilde{\mathcal{N}})$ to $RC_p(\tilde{\mathcal{N}})$. Let ϕ be a Schwarz function from \mathbb{R}^n to \mathbb{R} such that $\mathbf{1}_{\mathcal{A}_0}(x, y) \leq \phi(x - y) \leq \mathbf{1}_{|x-y| > 1/2}$ for all $x, y \in \mathbb{R}^n$. Let U be a smooth function from \mathbb{S}^{n-1} to the unit circle of \mathbb{C} with average 0. Such U exists, take for example $U_\lambda(x) = e^{2i \arctan(\lambda x_1)}$. Since \arctan is an odd function, the average of U_λ is real. Now, note that U_λ goes to -1 when λ goes to ∞ and to 1 when $\lambda = 0$ so there must exist a suitable U_λ . Extend U to \mathbb{R}^n by $U(0) = 0$ and $U(x) = U(x/|x|)$ otherwise. Write,

$$\begin{aligned} r_{k,s}(x, y) &= \mathbf{1}_{\mathcal{A}_k}(x, y) \frac{K(x) 2^{-(k+s)\gamma}}{I_{n,\gamma} |x - y|^{n+\gamma}} \\ &= \frac{K(x) 2^{-s\gamma}}{I_{n,\gamma}} \cdot \mathbf{1}_{\mathcal{A}_k}(x, y) \cdot \frac{\overline{U}(x - y) \phi(2^k(x - y))}{|2^k(x - y)|^\gamma} \cdot \frac{U(x - y)}{|x - y|^n} \end{aligned}$$

Denote by M_K the multiplication by $\frac{K(\cdot) 2^{-s\gamma}}{I_{n,\gamma}}$, M_K is bounded (see section 2.3). Define:

$$F(x) = \frac{\overline{U}(x) \phi(x)}{|x|^\gamma}.$$

Since ϕ is zero in a neighbourhood of the origin, F has no singularity and is a Schwarz function. So F is the Fourier transform of an L_1 function \widehat{F} . Note also that $\mathbf{1}_{\mathcal{A}_k} = \mathbf{1}_{x \in 5Q_{y,k}} - \mathbf{1}_{x \in 3Q_{y,k}}$, we will only prove the boundedness replacing $\mathbf{1}_{\mathcal{A}_k}$ by $\mathbf{1}_{x \in 5Q_{y,k}}$ since $\mathbf{1}_{x \in 3Q_{y,k}}$ is similar, denote the associated operator B'_i . Define also:

$$h(x) = \frac{U(x)}{|x|^n}$$

and H the convolution operator associated to h . Since U has 0 average, H is bounded from $RC_p(\tilde{\mathcal{N}})$ to $RC_p(\tilde{\mathcal{N}})$ for any $p \in (1, \infty)$. Indeed, H can then be written as an average of directionnal Hilbert operators (see [2]). Using notations from the previous proposition and lemma:

$$\begin{aligned} \mathbf{1}_{x \in 5Q_{y,k}} F(2^k(x-y))h(x-y) &= \int_{\Omega} l_k(x, \omega) F(2^k(x-y))h(x-y)g_k(y, \omega) d\omega \\ &= \int_{\Omega} \int_{\mathbb{R}^n} \widehat{F}(t) l_k(x, \omega) e^{2^k i t x} h(x-y) e^{-2^k i t y} g_k(y, \omega) dt d\omega. \end{aligned}$$

We will now translate this equality in terms of operators. Take again L and G from the previous part, the diagonal operators associated to the multiplication by $(l_k)_{k \in \mathbb{Z}}$ and $(g_k)_{k \in \mathbb{Z}}$, M_t defined by $M_t x = (e^{2^k i t x_k})_{k \in \mathbb{Z}}$, \mathcal{E} the expectation from $RC_p(\widetilde{\mathcal{N}})$ to $RC_p(\mathcal{N})$, and Δ such that $\Delta x = (\Delta_{k-s-i} x_k)_{k \in \mathbb{Z}}$. Then,

$$B'_i = M_K \int_{\mathbb{R}^n} \widehat{F}(t) \Delta \mathcal{E} L M_t H M_{-t} G dt.$$

As we have seen in the previous proof, M_t , M_{-t} , L and G are bounded by unconditionality of $RC_p(\widetilde{\mathcal{N}})$. More precisely, $\|M_t\|_{\mathcal{B}(RC_p(\widetilde{\mathcal{N}}))} = \|M_{-t}\|_{\mathcal{B}(RC_p(\widetilde{\mathcal{N}}))} = \|G\|_{\mathcal{B}(RC_p(\widetilde{\mathcal{N}}))} = 1$ and $\|L\|_{\mathcal{B}(RC_p(\widetilde{\mathcal{N}}))} = 5^n$. Δ is bounded thanks to Stein's inequality and we have constructed F and H such that \widehat{F} is L_1 and H is bounded on $RC_p(\widetilde{\mathcal{N}})$. Consequently,

$$\|B_i\|_{\mathcal{B}(L_p(\mathcal{N}))} \lesssim 2 \|\Delta\|_{\mathcal{B}(RC_p(\mathcal{N}))} 5^n \|\widehat{F}\|_1 \|H\|_{\mathcal{B}(RC_p(\mathcal{N}))}. \quad \square$$

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